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An introduction to the fixed-circle problem with metric-preserving functions

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Abstract. There are many studies in the literature about both the concept of a metric-preserving function and the fixed-circle problem. In this paper, we obtain new fixed-circle theorems by using these two concepts together. We support the theoretical results we obtained with necessary examples. The importance of this study is that it contains the first fixed-circle theorems obtained using the metric-preserving function concept.

1. Introduction and preliminaries

Let X be a nonempty set and the function $d : X \times X \to [0, \infty)$ be a function satisfying the followings:

(M1) $d(x, y) = 0 \Longleftrightarrow x = y$ for all $x, y \in X$,
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(M3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then the function d is a metric on X and the pair (X, d) is a metric space.

The concept of a metric space is one of the concepts that are fundamental to the study of different subjects in many areas of mathematics. It has an important place in the examination of both topological studies and application-oriented studies. Metric fixed-point theory is one of the most common research topics on metric spaces.

So what is a fixed point?

Fixed Point: Let X be a nonempty set and $T : X \to X$ be a self-mapping. If $Tx = x$ for some $x \in X$, then x is a fixed point of T. The fixed-point set of T is denoted by $Fix(T)$.

It should be noted that the number of elements of the fixed point set of a function may vary. For example, the function "*sin*" has three fixed points and the function "*cos*" has a unique fixed point.

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After this example, when the number of elements in the fixed point set of a given function is more than one, it is important to make a geometric interpretation of this fixed point set. The fixed-circle problem began to be studied under this perspective [18].

Let's remember the definitions of circle and fixed circle in metric spaces, which are necessary to study the fixed-circle problem.

Circle: Let (X, d) be a metric space. A circle $C(x_0, r)$ of radius r centered at x_0 is defined by

$$C(x_0, r) = \{x \in X : d(x, x_0) = r\}.$$

Fixed Circle: Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. If $Tx = x$ for $x \in C(x_0, r)$, then $C(x_0, r)$ is a fixed circle of T [18].

This problem continues to be studied from different perspectives. Let's give a few examples of these studies: In [6], the fixed-circle problem was generalized the fixed-ellipse problem and some fixed-ellipse results were obtained. In [8], new fixed-circle results were given with different auxiliary functions. In [9] and [10], some fixed-circle theorems were proved in fuzzy metric spaces and G-metric spaces, respectively. In [16] and [22], some common fixed-circle theorems were presented in metric and S-metric spaces. In [17], the discontinuity problem at fixed point and the fixed-circle problem were combined to prove the new results. In [19], Özgür investigated some fixed-disc results via simulation functions. In [21], Taş proved bilateral type fixed-circle theorems with an application to the rectified linear unit activation functions.

Another important subject studied using the concept of metric space is the concept of metric-preserving function as seen the following definition:

Metric-Preserving Function: Let (X, d) be a metric space and $f : [0, \infty) \to [0, \infty)$ be a function. Then f is called a metric-preserving function if $f \circ d$ is a metric on X [2].

Using the notion of metric-preserving functions, studies are ongoing: In [5], some properties of metric-preserving functions were given. In [11] and [12], the notion of a b-metric-preserving function was introduced and some basic relations were given. In [13], some "Pasting Lemmas" were proved for b-metric-preserving functions. In [14], the concept of an extended b-metric-preserving function was given and some properties were obtained. In [20], new fixed-point results were proved using the notion of a metric-preserving function.

By the above motivation, in this paper, we introduce some new circle notions using the concept of metric preserving functions with different examples. We prove new fixed-circle theorems using these new notions. We support our results with necessary examples.

2. The notion of a circle via metric-preserving functions

In this section, we introduce the notions of some circles according to the metric-preserving functions. To do this, let (X, d) be a metric space, $C(x_0, r)$ be a circle on X and $f : [0, \infty) \to [0, \infty)$ be a metric-preserving function.

Assume that the metric function $f \circ d : X \times X \to [0, \infty)$ is denoted by d_f, that is,

$$f_d = f \circ d$$

and the radius r_f is denoted by

$$r_f = f(r).$$

Now we define the following definitions:

Definition 1. A circle $C(x_0, r_f)$ of radius r_f centered at x_0 in a metric space (X, d) is the set of all points of X of equal to r_f from x_0, that is,

$$C(x_0, r_f) = \{x \in X : d(x, x_0) = r_f\}.$$

Definition 2. A circle $C_f(x_0, r)$ of radius r centered at x_0 in a metric space (X, d_f) is the set of all points of X of equal to r from x_0, that is,

$$C_f(x_0, r) = \{x \in X : d_f(x, x_0) = r\}.$$

Definition 3. A circle $C(f(x_0), r)$ of radius r centered at $f(x_0)$ in a metric space (X, d) is the set of all points of X of equal to r from $f(x_0)$, that is,

$$C(f(x_0), r) = \{x \in X : d(x, f(x_0)) = r\}.$$

Definition 4. A circle $C(f(x_0), r_f)$ of radius r_f centered at $f(x_0)$ in a metric space (X, d) is the set of all points of X of equal to r_f from $f(x_0)$, that is,

$$C(f(x_0), r_f) = \{x \in X : d(x, f(x_0)) = r_f\}.$$

Definition 5. A circle $C_f(x_0, r_f)$ of radius r_f centered at x_0 in a metric space (X, d_f) is the set of all points of X of equal to r_f from x_0, that is,

$$C_f(x_0, r_f) = \{x \in X : d_f(x, x_0) = r_f\}.$$

Some comparisons of these notations are give as follows:

- If $x_0 \in Fix(f)$ then the circles $C(x_0, r_f)$ and $C(f(x_0), r_f)$ are coincide.
- If $x_0 \in Fix(f)$ then the circles $C(x_0, r)$ and $C(f(x_0), r)$ are coincide.
- If $r = r_f$ then the circles $C(x_0, r)$ and $C(x_0, r_f)$ are coincide.
- If $r = r_f$ then the circles $C_f(x_0, r)$ and $C_f(x_0, r_f)$ are coincide.
- If the metric-preserving function f is defined as

  $$f(x) = ax,$$

  for all $x \in [0, \infty)$ where $a > 0$, then he circles $C(x_0, r)$ and $C_f(x_0, r_f)$ are coincide (as seen in the following examples).

We give the following examples to show the difference of these circle notions.

Example 1. Let $(\mathbb{R}, d)$ be a usual metric space with the metric $d(x, y) = |x - y|$ and $C_{0,1} = \{-1, 1\}$ be a unit circle. Let us define the metric-preserving function $f : [0, \infty) \to [0, \infty)$ as

$$f(x) = 2x,$$

for all $x \in [0, \infty)$. Then we get

$$r_f = f(r) = f(1) = 2$$

and

$$f(0) = 0.$$

Hence using the above definitions, we obtain the following circles:

$$\begin{aligned} C(x_{0,}r_f) &= C(0,2) = \{x \in \mathbb{R} : d(x,0) = 2\} \\ &= \{x \in \mathbb{R} : |x - 0| = 2\} \\ &= \{x \in \mathbb{R} : |x| = 2\} \\ &= \{-2, 2\}, \end{aligned}$$

$$\begin{aligned} C_f(x_0, r) &= C_f(0,1) = \{x \in \mathbb{R} : d_f(x,0) = 1\} \\ &= \{x \in \mathbb{R} : 2\,|x - 0| = 1\} \\ &= \left\{x \in \mathbb{R} : |x| = \frac{1}{2}\right\} \\ &= \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \end{aligned}$$

$$\begin{aligned} C(f(x_0), r) &= C(0,1) = \{x \in \mathbb{R} : d(x,0) = 1\} \\ &= \{x \in \mathbb{R} : |x - 0| = 1\} \\ &= \{x \in \mathbb{R} : |x| = 1\} \\ &= \{-1, 1\}, \end{aligned}$$

$$\begin{aligned} C(f(x_0), r_f) &= C(0,2) = \{x \in \mathbb{R} : d(x,0) = 2\} \\ &= \{x \in \mathbb{R} : |x - 0| = 2\} \\ &= \{x \in \mathbb{R} : |x| = 2\} \\ &= \{-2, 2\} \end{aligned}$$

and

$$\begin{aligned} C_f(x_0, r_f) &= C_f(0,2) = \{x \in \mathbb{R} : d_f(x,0) = 2\} \\ &= \{x \in \mathbb{R} : 2\,|x - 0| = 2\} \\ &= \{x \in \mathbb{R} : |x| = 1\} \\ &= \{-1, 1\}. \end{aligned}$$

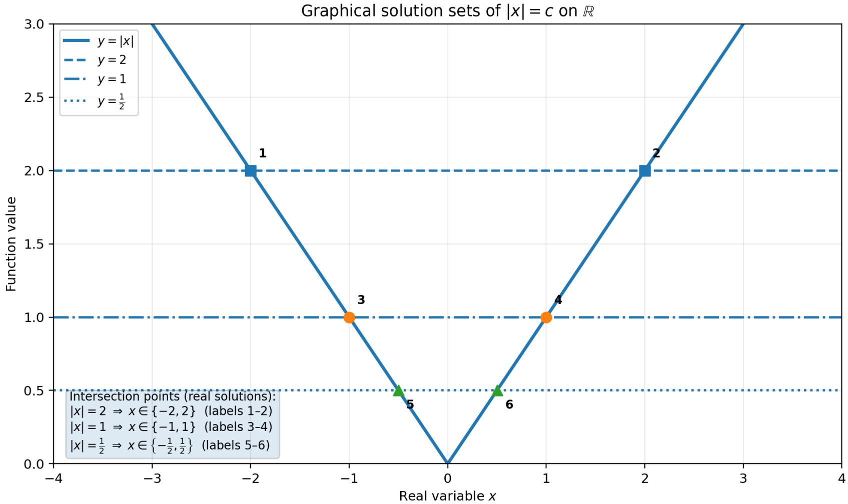

FIGURE 1. Graphical representation of the resulting circles in Example 1.

Example 2. Let $(\mathbb{R}, d)$ be a usual metric space and

$$C(2,1) = \{x \in \mathbb{R} : |x-2| = 1\} = \{1,3\}$$

be a circle. Let us define the metric-preserving function $f : [0,\infty) \to [0,\infty)$ as

$$f(x) = \frac{x}{3},$$

for all $x \in [0,\infty)$. Then we get

$$r_f = f(r) = f(1) = \frac{1}{3}$$

and

$$f(2) = \frac{2}{3}.$$

Hence, using the above definitions, we obtain the following circles:

$$\begin{aligned} C(x_0, r_f) &= C\left(2, \frac{1}{3}\right) = \left\{x \in \mathbb{R} : d(x,2) = \frac{1}{3}\right\} \\ &= \left\{x \in \mathbb{R} : |x-2| = \frac{1}{3}\right\} \\ &= \left\{\frac{5}{3}, \frac{7}{3}\right\}, \end{aligned}$$

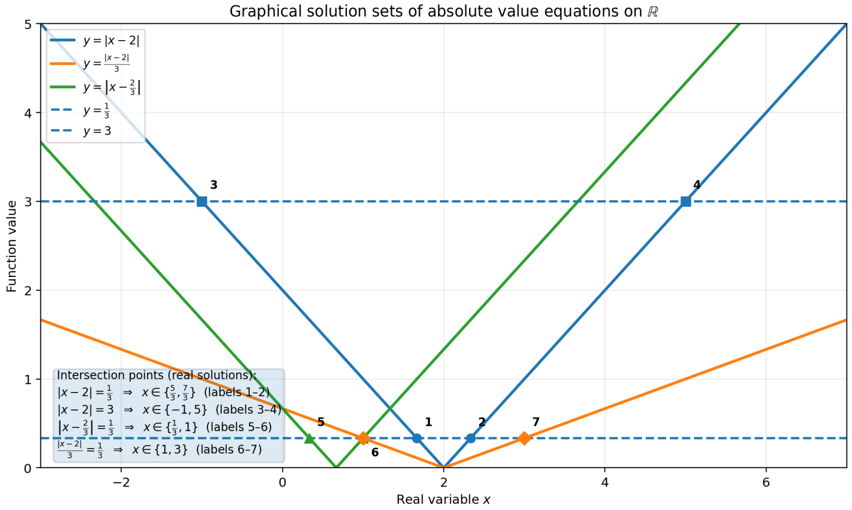

FIGURE 2. Graphical representation of the resulting circles in Example 2.

$$
\begin{aligned}
C_f(x_0, r) &= C_f(2,1) = \{x \in \mathbb{R} : d_f(x,2) = 1\} \\
&= \left\{x \in \mathbb{R} : \frac{1}{3}|x-2| = 1\right\} \\
&= \{x \in \mathbb{R} : |x-2| = 3\} \\
&= \{-1, 5\},
\end{aligned}
$$

$$
\begin{aligned}
C(f(x_0), r) &= C(f(2), 1) = C\left(\frac{2}{3}, 1\right) \\
&= \left\{x \in \mathbb{R} : d\left(x, \frac{2}{3}\right) = 1\right\} \\
&= \left\{x \in \mathbb{R} : \left|x - \frac{2}{3}\right| = 1\right\} \\
&= \left\{-\frac{1}{3}, \frac{5}{3}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C(f(x_0), r_f) &= C(f(2), r_f) = C\left(\frac{2}{3}, \frac{1}{3}\right) \\
&= \left\{x \in \mathbb{R} : \left|x - \frac{2}{3}\right| = \frac{1}{3}\right\} \\
&= \left\{\frac{1}{3}, 1\right\}
\end{aligned}
$$

and

$$\begin{aligned} C_f(x_0, r_f) &= C_f\left(2, \frac{1}{3}\right) = \left\{x \in \mathbb{R} : d_f\left(x, 2\right) = \frac{1}{3}\right\} \\ &= \left\{x \in \mathbb{R} : \frac{|x-2|}{3} = \frac{1}{3}\right\} \\ &= \{1, 3\}. \end{aligned}$$

Example 3. Let $X = \mathbb{R}$ be a discrete metric space with the discrete metric

$$d_A = \begin{cases} 0, & x = y; \\ 1, & x \neq y; \end{cases}$$

for all $x, y \in \mathbb{R}$ and $C_{0,\frac{1}{2}}$ be a circle on (X, d_A). Let us define the metric-preserving function $f : [0, \infty) \to [0, \infty)$ as

$$f(x) = \begin{cases} 2x, & x = \frac{1}{2}; \\ x, & \text{otherwise}; \end{cases}$$

for all $x \in [0, \infty)$. Then we get

$$C_{0,\frac{1}{2}} = \left\{x \in \mathbb{R} : d_A\left(x, 0\right) = \frac{1}{2}\right\} = \emptyset,$$

$$f\left(\frac{1}{2}\right) = 1 = r_f,$$

$$d_{A_f} = f \circ d_A = \begin{cases} 0, & x = y; \\ 1, & x \neq y; \end{cases}$$

and

$$C_f\left(0, 1\right) = \mathbb{R} - \{0\}.$$

We notice that although the circle $C_{0,\frac{1}{2}}$ is an empty set according to the discrete metric d_A, the circle $C_f\left(0,1\right)$ has infinite elements according to the metric d_{A_f}. In this case, it is possible to obtain a circle with infinite elements by using the concept of metric-preserving function.

3. Some fixed-circle theorems

In this section, we prove some fixed-circle theorems related to metric-preserving functions.

Theorem 1. *Let (X, d) be a metric space, f be a metric-preserving function and $C(x_0, r)$ be any circle on X. Let us define the mapping $\varphi_f : X \to [0, \infty)$ as*

(1) $$\varphi_f(x) = f \circ d(x, x_0),$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

(Cf_1) $f \circ d(x, Tx) \leq \varphi f(x) - \varphi f(Tx)$,
(Cf_2) $f \circ d(Tx, x_0) \geq r$,

for each $x \in C_f(x_{0,}r)$, *then* $C_f(x_{0,}r)$ *is a fixed circle of* T.

Proof. Let $x \in C_f(x_0, r)$ be any point. Using the condition (Cf_1) and the definition of φ_f, we get

$$f \circ d(x, Tx) \leq f \circ d(x, x_0) - f \circ d(Tx, x_0) = r - f \circ d(Tx, x_0), \tag{2}$$

Using the inequalities (Cf_2) and (2), it should be

$$f \circ d(Tx, x_0) = r.$$

Then, we have

$$f \circ d(x, Tx) \leq 0,$$

that is, since $f \circ d$ is a metric function, we get

$$f \circ d(x, Tx) = 0 \quad \Longrightarrow \quad x = Tx.$$

Consequently, $C_f(x_{0,}r)$ is a fixed circle of T. □

Example 4. Let (X, d) be a metric space and f be a metric-preserving function. Let us consider a circle $C_f(x_0, r)$ and define the self mapping $T : X \to X$ as

$$T_1 x = \begin{cases} x, & x \in C_f(x_0, r); \\ x_0, & \text{otherwise}; \end{cases}$$

for all $x \in X$. Then the self-mapping T satisfies the conditions (Cf_1), (Cf_2) and so T fixes the circle $C_f(x_{0,}r)$.

Example 5. Let $(\mathbb{R}, d)$ be a usual metric space and the function $f : [0, \infty) \to [0, \infty)$ be defined as

$$f_x = \alpha x, \quad \alpha > 1, \tag{3}$$

for all $x \in [0, \infty)$. Let us consider the unit circle $C(0, 1) = \{-1, 1\}$ and the self-mapping $T : \mathbb{R} \to \mathbb{R}$ defined as

$$T_2 x = \begin{cases} 2x - \frac{1}{\alpha}, & x = \frac{1}{\alpha}; \\ -|x|, & x = -\frac{1}{\alpha}; \\ 0, & \text{otherwise}; \end{cases}$$

for all $x \in \mathbb{R}$. Hence, T satisfies the conditions (Cf_1) and (Cf_2) for each $x \in C_f(0, 1)$. Indeed, we have the following two cases:

Case 1: Let $x = \frac{1}{\alpha}$. Then we have

$$T_2 \frac{1}{\alpha} = \frac{1}{\alpha}.$$

(Cf_1) For $x = \frac{1}{\alpha}$, we get

$$\begin{aligned} f \circ d\left(\frac{1}{\alpha}, T_2\frac{1}{\alpha}\right) &= f \circ d\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) = 0 \\ &\leq \varphi_f\left(\frac{1}{\alpha}\right) - \varphi_f\left(T_2\frac{1}{\alpha}\right) \\ &= f \circ d\left(\frac{1}{\alpha}, 0\right) - f \circ d\left(T_2\frac{1}{\alpha}, 0\right) = 0. \end{aligned}$$

(Cf_2) For $x = \frac{1}{\alpha}$, we get

$$\begin{aligned} f \circ d\left(T_2\frac{1}{\alpha}, 0\right) &= f \circ d\left(\frac{1}{\alpha}, 0\right) = f\left(\frac{1}{\alpha}\right) \\ &= \alpha \cdot \frac{1}{\alpha} = 1 \geq 1. \end{aligned}$$

Case 2: Let $x = -\frac{1}{\alpha}$. Then we have

$$T_2\left(-\frac{1}{\alpha}\right) = -\frac{1}{\alpha}.$$

(Cf_1) For $x = -\frac{1}{\alpha}$, we get

$$\begin{aligned} f \circ d\left(-\frac{1}{\alpha}, T_2\left(-\frac{1}{\alpha}\right)\right) &= f \circ d\left(-\frac{1}{\alpha}, -\frac{1}{\alpha}\right) = 0 \\ &\leq \varphi_f\left(-\frac{1}{\alpha}\right) - \varphi_f\left(T_2\left(-\frac{1}{\alpha}\right)\right) \\ &= f \circ d\left(-\frac{1}{\alpha}, 0\right) - f \circ d\left(T_2\left(-\frac{1}{\alpha}\right), 0\right) = 0. \end{aligned}$$

(Cf_2) For $x = -\frac{1}{\alpha}$, we get

$$f \circ d\left(T_2\left(-\frac{1}{\alpha}\right), 0\right) = f \circ d\left(-\frac{1}{\alpha}, 0\right) = f\left(-\frac{1}{\alpha}\right) = -\alpha\left(-\frac{1}{\alpha}\right) = 1.$$

Consequently, $C_f(0,1)$ is fixed circle of T_2. Notice that T_2 fixes the circle $C_f(0,1)$. But T_2 does not fix the unit circle $C(0,1)$. Also any self-mapping T does not have to fix the center of circle. For example, let us define the self-mapping $T_3 : X \to X$ as

$$T_3x = \begin{cases} x, & x \in C_f(x_0, r); \\ x^*, & \text{otherwise}; \end{cases}$$

for all $x \in \mathbb{R}$, where

$$f \circ d(x^*, x_0) > r.$$

Then T_3 fixed the circle $C_f(x_0, r)$ but T_3 does not fix the centre of circle.

Theorem 2. *Let (X, d) be a metric-space, f be an increasing metric-preserving function and $C(x_0, r)$ be any circle on X. Let the mapping φ_f be defined as in (3). If there exists a self-mapping $T : X \to X$ a satisfying*

(Cf_1') $d(x, Tx) \leq \varphi_f(x) - \varphi_f(Tx)$,

(Cf_2') $d(Tx, x_0) \geq r_f$,

for each $x \in C(x_0, r_f)$, then the circle $C(x_0, r_f)$ is a fixed circle of T.

Proof. Let $C(x_0, r_f)$ be any point. Using the condition (Cf_1') and the definition of φ_f, we get

$$\begin{aligned} d(x, Tx) &\leq \varphi_f(x) - \varphi_f(Tx) \\ &= f \circ d(x, x_0) - f \circ d(Tx, x_0) \\ &= f(r_f) - f(d(Tx, x_0)). \end{aligned}$$

By (Cf_2'), it should be

$$d(Tx, x_0) = r_f$$

and so we obtain

$$d(x, Tx) = 0 \quad \Longrightarrow \quad Tx = x.$$

Consequently, $C(x_0, r_f)$ is a fixed circle of T. □

Example 6. Let $(\mathbb{R}, d)$ be a usual metric space and the function $f : [0, \infty) \to [0, \infty)$ be defined as in (3). Let us consider the circle

$$C(0, 2) = \{-2, 2\}$$

and the self-mapping $T_4 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_4 x = \begin{cases} 3x - 4\alpha, & x = 2\alpha; \\ x, & x \in \{-2\alpha, 0\}; \\ \alpha, & \text{otherwise}; \end{cases}$$

for all $x \in \mathbb{R}$. Hence, T_4 satisfies the conditions (Cf_1') and (Cf_2') for each $x \in C(0, 2\alpha) = \{-2\alpha, 2\alpha\}$. Indeed, we have the following two cases:

Case 1: Let $x = 2\alpha$. Then we have

$$T_4(2\alpha) = 2\alpha.$$

(Cf_1') For $x = 2\alpha$, we get

$$\begin{aligned} d(2\alpha, T_4(2\alpha)) &\leq \varphi_f(2\alpha) - \varphi_f(T_4(2\alpha)) \\ &= f \circ d(2\alpha, 0) - f \circ d(T_4(2\alpha), 0) = 0. \end{aligned}$$

(Cf_2') For $x = 2\alpha$, we get

$$d(T_4(2\alpha), 0) = d(2\alpha, 0) = 2\alpha = r_f.$$

Case 2: Let $x = -2\alpha$. Then we have

$$T_4(-2\alpha) = -2\alpha.$$

($\mathrm{Cf}_1^{'}$) For $x = -2\alpha$, we get

$$\begin{aligned} d(-2\alpha, T_4(-2\alpha)) &\leq \varphi_f(-2\alpha) - \varphi_f(T_4(-2\alpha)) \\ &= f \circ d(-2\alpha, 0) - f \circ d(T_4(-2\alpha), 0) = 0. \end{aligned}$$

($\mathrm{Cf}_2^{'}$) For $x = -2\alpha$, we get

$$d(T_4(-2\alpha), 0) = d(-2\alpha, 0) = 2\alpha = r_f.$$

Consequently, $C(0, 2\alpha)$ is a fixed circle of T_4.

Theorem 3. *Let (X, d)be a metric space, f be a metric-preserving function and $C(x_0, r)$ be any circle on X. Let us define the mapping φ_f be defined as (1). If there exists a self-mapping $T : X \to X$ satisfying*

($\mathrm{Cf}_1^{''}$) $d(x, Tx) \leq \varphi f(x) - \varphi f(Tx)$,

($\mathrm{Cf}_2^{''}$) $f \circ d(Tx, x_0) \geq r_f$,

for each $x \in C_f(x_0, r_f)$, then the circle $C_f(x_0, r_f)$ is a fixed circle of T.

Proof. Let $x \in C_f(x_0, r_f)$ be any point. Using the condition ($\mathrm{Cf}_1^{''}$) and the definition of φ_f, we get

$$f \circ d(x, x_0) = r_f = f(r)$$

and

$$\begin{aligned} d(x, Tx) &\leq \varphi_f(x) - \varphi_f(Tx) \\ &= f \circ d(x, x_0) - f \circ d(Tx - x_0) \\ &= r_f - f \circ d(Tx, x_0). \end{aligned}$$

By ($\mathrm{Cf}_2^{''}$), it should be

$$f \circ d(Tx, x_0) = r_f$$

and so we obtain

$$d(x, Tx) = 0 \quad \Longrightarrow \quad Tx = x.$$

Consequently, $C_f(x_0, r_f)$ is a fixed circle of T. □

Example 7. Let $(\mathbb{R}, d_A)$ be a discrete metric space with the metric function defined as

$$d_A(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y; \end{cases}$$

for all $x, y \in \mathbb{R}$ and the function $f : [0, \infty) \to [0, \infty)$ be defined as

$$f(x) = \begin{cases} \alpha x, & x = \frac{1}{\alpha}; \\ x, & \text{otherwise}; \end{cases}$$

for all $x \in [0, \infty)$ where $\alpha \in \{n : 2 \leq n \leq 100\}$. Let us consider the circle

$$C\left(0, \frac{1}{\alpha}\right) = \left\{x \in \mathbb{R} : d_A(x, 0) = \frac{1}{\alpha}\right\} = \emptyset$$

and the self mapping $T_5 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_5 x = \begin{cases} x, & x \in \mathbb{R} - \{0\}; \\ \alpha, & x = 0; \end{cases}$$

for all $x \in \mathbb{R}$. Notice that the function f is not a one to one function. Indeed, for $x = 1$, $y = \frac{1}{\alpha}$, we have $x \neq y$ and $f(x) = 1 = f(y)$. Also, we get

$$f \circ d_A(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y; \end{cases}$$

and

$$C_f(0, 1) = \{x \in \mathbb{R} : f \circ d_A(x, 0) = 1\} = \mathbb{R} - \{0\}.$$

It can be easily seen that the self-mapping T_5 satisfies the conditions $(\mathrm{Cf}_1^{''})$ and $(\mathrm{Cf}_2^{''})$ for each $x \in C_f(0, 1)$. Consequently, $C_f(0, 1)$ is a fixed circle of T_5.

Theorem 4. *Let (X, d) be a metric space, f be a metric-preserving function and $C(x_0, r)$ be a any circle on X. Let us define the mapping $\varphi_f^* : X \to [0, \infty)$ as*

$$\varphi_f^*(x) = d(x, f(x_0)). \tag{4}$$

If there exists a self mapping $T : X \to X$ satisfying
$(\mathrm{Cf}_1^{'''})$ *$d(x, Tx) \leq \varphi_f^*(x) - \varphi_f^*(Tx)$,*
$(\mathrm{Cf}_2^{'''})$ *$d(Tx, f(x_0)) \geq r$,*
for each $x \in C(f(x_0), r)$, then the circle $C(f(x_0), r)$ is a fixed circle of T.

Proof. Let $x \in C(f(x_0), r)$ be any point. Using the condition $(\mathrm{Cf}_1^{'''})$ and the definition of φ_f^*, we get

$$d(x, f(x_0)) = r$$

and

$$\begin{aligned} d(x, Tx) &\leq \varphi_f^*(x) - \varphi_f^*(Tx) \\ &= d(x, f(x_0)) - d(Tx, f(x_0)) \\ &= r - d(Tx, f(x_0)). \end{aligned}$$

By $(\mathrm{Cf}_2^{'''})$, it should be

$$d(Tx, f(x_0)) = r_f$$

and so we obtain

$$d(x, Tx) = 0 \implies Tx = x.$$

Consequently, $C(f(x_0), r)$ is a fixed circle of T. □

Example 8. Let $(\mathbb{R}, d)$ be a usual metric space and the function $f : [0, \infty) \to [0, \infty)$ be defined as

$$f(x) = 2x,$$

for all $x \in [0, \infty)$. Let us consider the circle

$$C(1, 1) = \{0, 2\}$$

and the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_6(x) = \begin{cases} x, & x \in \{1, 3\}; \\ 2, & \text{otherwise}; \end{cases}$$

for all $x \in \mathbb{R}$. Hence, it is clear that T_6 satisfies the conditions $(\mathrm{Cf}_1^{'''})$ and $(\mathrm{Cf}_2^{'''})$ for each $x \in C(2, 1) = \{1, 3\}$.

Theorem 5. *Let (X, d) be a metric space, f be a metric-preserving function and $C(x_0, r)$ be a any circle on X. Let us define the mapping $\varphi_f^* : X \to [0, \infty)$ be defined as in (4). If there exists a self mapping $T : X \to X$ satisfying*

(Cf_1^{iv}) $d(x, Tx) \leq \varphi_f^*(x) - \varphi_f^*(Tx)$,

(Cf_2^{iv}) $d(Tx, f(x_0)) \geq r_f$,

for each $x \in C(f(x_0), r_f)$, then the circle $C(f(x_0), r_f)$ is a fixed circle of T.

Proof. Let $x \in C(f(x_0), r_f)$ be any point. Using the condition (Cf_1^{iv}) and the definition of φ_f^*, we get

$$d(x, f(x_0)) = r_f$$

and

$$\begin{aligned} d(x, Tx) &\leq \varphi_f^*(x) - \varphi_f^*(Tx) \\ &= d(x, f(x_0)) - d(Tx, f(x_0)) \\ &= r_f - d(Tx, f(x_0)). \end{aligned}$$

By (Cf_2^{iv}), it should be

$$d(Tx, f(x_0)) = r_f$$

and so we obtain

$$d(x, Tx) = 0 \quad \Longrightarrow \quad Tx = x.$$

Consequently, $C(f(x_0), r_f)$ is a fixed circle of T. □

Example 9. Let $(\mathbb{R}, d)$ be a usual metric space and the function $f : [0, \infty) \to [0, \infty)$ be defined as

$$f(x) = 5x,$$

for all $x \in [0, \infty)$. Let us consider the circle

$$C(1, 2) = \{-1, 3\}$$

and the self-mapping $T_7 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_7(x) = \begin{cases} x, & x \in \{-5, 15\}; \\ 5, & \text{otherwise}; \end{cases}$$

for all $x \in \mathbb{R}$. Hence, it is obvious that T_7 satisfies the conditions (Cf_1^{iv}) and (Cf_2^{iv}) for each $x \in C(5, 10) = \{-5, 15\}$.

Consequently, the significance of this study can be summarized as follows:

The fixed-circle problem has emerged as a geometric counterpart of fixed point theory, and several existing works obtain fixed-circle results by adapting classical contraction-type assumptions (e.g., Banach-type [1], Kannan-type [7], Chatterjea-type [3], Ciric-type [4], Meir-Keeler-type [15], and various interpolative or rational contractions) to the circle setting. In most of these contributions, the circle is characterized through a specific "radius function" (often a distance from a prescribed center) and the fixed-circle conclusions are derived from contractive inequalities imposed directly on the underlying metric d and the self-mapping T. Hence, the novelty in many earlier studies lies primarily in the type of contraction used, while the geometric structure (how the circle condition is encoded) typically remains tied to the original metric.

In contrast, the main contribution of the present paper is to shift the focus from the contraction family alone to the metric transformation layer by incorporating metric-preserving functions. Rather than working only with d, we construct new circle notions by applying metric-preserving functions to the metric and by combining these transformations with circle-defining expressions. This viewpoint yields circle concepts that are not immediate reformulations of the classical circle condition, and it allows the fixed-circle problem to be studied under a broader class of "metric-compatible" deformations. Consequently, our fixed-circle theorems are obtained through hypotheses formulated in terms of these new notions, which can recover standard fixed-circle statements as special cases when the metric-preserving function reduces to the identity, while also generating genuinely different scenarios when nontrivial metric-preserving functions are employed.

Another difference from earlier works is methodological: many existing results rely on a single geometric descriptor of a circle (e.g., a fixed center and radius measured by d), whereas our approach produces multiple circle notions stemming from different metric-preserving transformations and their interaction with the map T. This provides additional flexibility in constructing examples and in modeling situations where distances are altered through admissible transformations without losing metric structure. Finally, the paper emphasizes illustrative examples tailored to demonstrate that these notions are not merely cosmetic generalizations; they enlarge the scope of fixed-circle applicability and show how distinct metric-preserving functions lead to different fixed-circle behaviors even for maps acting on the same underlying set.

4. Conclusion and future works

In this study, we have investigated the fixed-circle problem through the lens of metric-preserving functions and established several new fixed-circle theorems under appropriate conditions. By combining these two well-studied concepts, we have introduced a novel methodological framework that broadens the scope of fixed-circle theory. The validity and applicability of the obtained results have been reinforced by illustrative examples, which demonstrate the effectiveness of the proposed approach.

The main significance of this work lies in the fact that it presents the first fixed-circle results derived by employing metric-preserving functions. This contribution not only enriches the existing literature on fixed-circle problems but also highlights the potential of metric-preserving functions as a powerful tool in geometric fixed point theory. We believe that the results obtained herein will stimulate further research in this direction and inspire new investigations into fixed geometric structures under alternative contraction-type conditions.

The results presented in this paper open several promising directions for future research. One natural extension of this study is to investigate fixed-circle problems under different types of metric-preserving functions and to explore whether similar results can be obtained in more general metric-like structures, such as partial metric spaces, b-metric spaces, or G-metric spaces.

Another potential research direction is the formulation of fixed-circle theorems by employing alternative contraction conditions combined with metric-preserving functions. In particular, studying interpolative, rational, or multivalued contractions within this framework may lead to new and meaningful generalizations.

Furthermore, extending the current approach to higher-dimensional geometric fixed structures, such as fixed spheres or fixed sets, could provide a deeper geometric understanding of fixed point theory. Finally, the applicability of the developed results to applied problems in nonlinear analysis and related fields may also be explored, thereby enhancing the practical relevance of the theoretical findings presented in this work.

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